Sampled-data reachability analysis using sensitivity and mixed-monotonicity

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Reachability analysis

Discrete-time system

$$x^+ = F(x),\ x \in \mathbb{R}^n$$

Continuous-time system $\dot{x} = f(x)$
System trajectories for initial state x_0

$$x(t) = \Phi(t; x_0), \quad \forall t \geq 0$$

Reachability analysis

Discrete-time system

$$x^+ = F(x),\ x \in \mathbb{R}^n$$

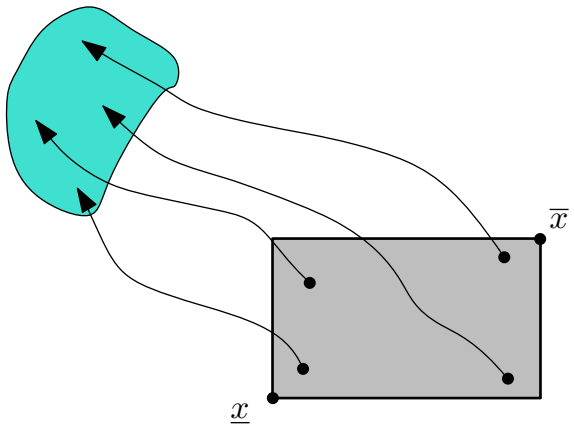
Reachable set $F([\underline{x}, \overline{x}])$
for initial conditions $x \in [\underline{x}, \overline{x}]$

Continuous-time system $\dot{x} = f(x)$
System trajectories for initial state x_0

$$x(t) = \Phi(t; x_0), \quad \forall t \geq 0$$

Reachable set at time $T \geq 0$
under initial conditions $x_0 \in [\underline{x}, \overline{x}]$

$$RS(T, [\underline{x}, \overline{x}]) = \{\Phi(T; x_0) \mid x_0 \in [\underline{x}, \overline{x}]\}$$



Sign-stable Jacobian - Illustration

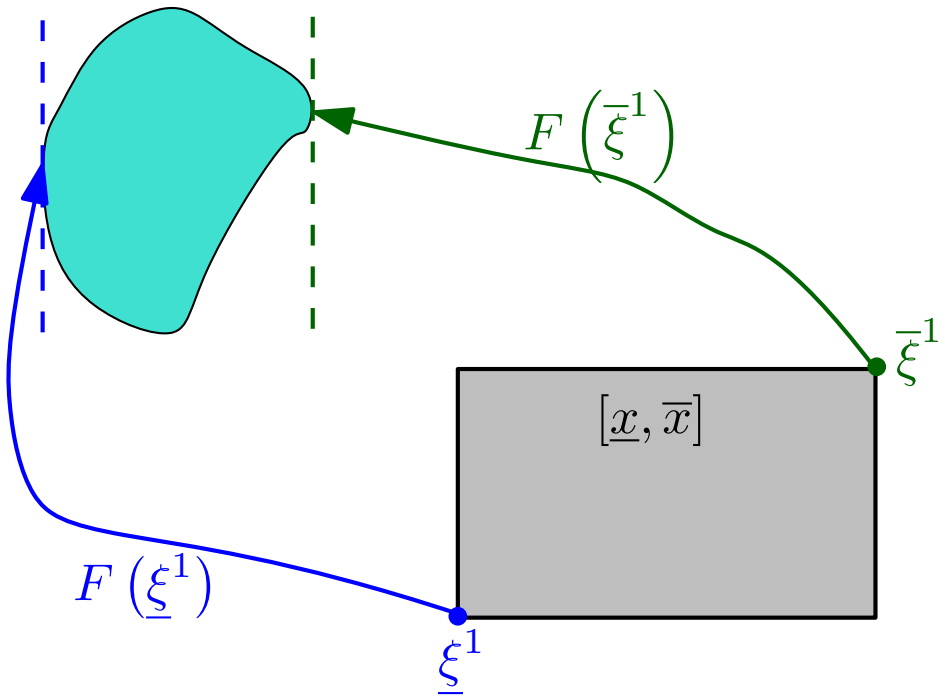
Assumption

$\frac{\partial F(x)}{\partial x}$ *is* **sign-stable** *over the set of states* $[\underline{x}, \overline{x}]$

$$sign\left(\frac{\partial F(x)}{\partial x}\right) = \begin{pmatrix} + & + \\ - & + \end{pmatrix}$$

Dimension 1:
Find $\underline{\xi}^1, \overline{\xi}^1 \in [\underline{x}, \overline{x}]$ such that
$F_1([\underline{x}, \overline{x}]) \subseteq [F_1(\underline{\xi}^1), F_1(\overline{\xi}^1)]$

$$\underline{\xi}^1 = \begin{pmatrix} \underline{x}_1 \\ \underline{x}_2 \end{pmatrix} \quad \overline{\xi}^1 = \begin{pmatrix} \overline{x}_1 \\ \overline{x}_2 \end{pmatrix}$$



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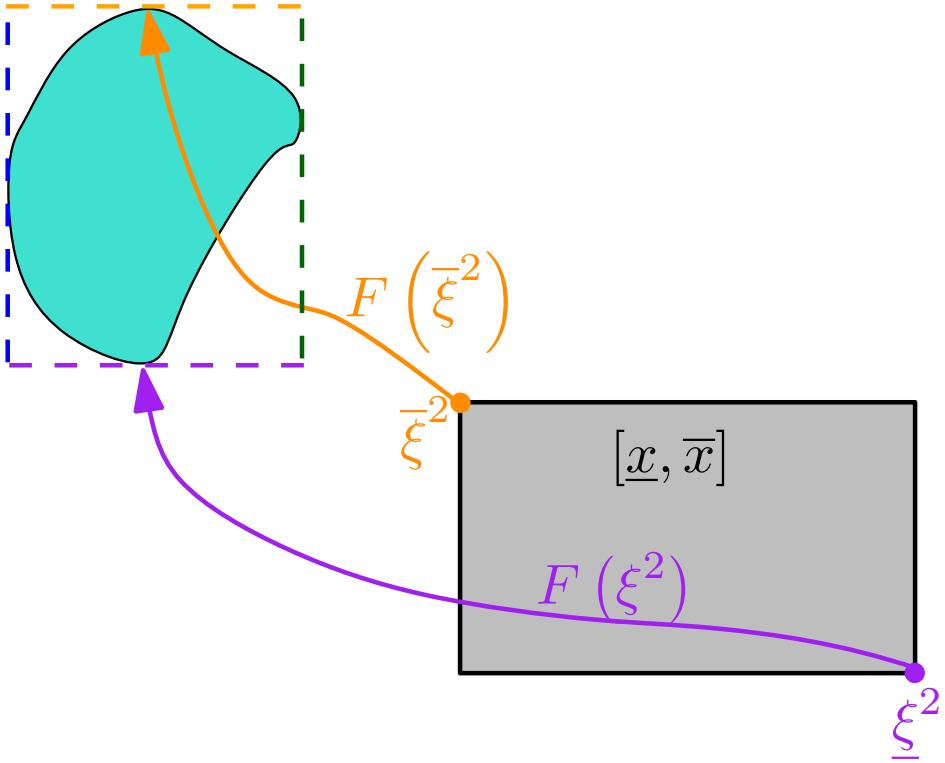
$$sign\left(\frac{\partial F(x)}{\partial x}\right) = \begin{pmatrix} + & + \\ - & + \end{pmatrix}$$

Dimension 2:
Find $\underline{\xi}^2, \overline{\xi}^2 \in [\underline{x}, \overline{x}]$ such that
$F_2([\underline{x}, \overline{x}]) \subseteq [F_2(\underline{\xi}^2), F_2(\overline{\xi}^2)]$

$$\underline{\xi}^2 = \begin{pmatrix} \overline{x}_1 \\ \underline{x}_2 \end{pmatrix} \quad \overline{\xi}^2 = \begin{pmatrix} \underline{x}_1 \\ \overline{x}_2 \end{pmatrix}$$



Tight over-approximation and **linear complexity** ($2n$ evaluations of F)

Bounded Jacobian

Assumption

$\frac{\partial F(x)}{\partial x}$ *is* **bounded**: $\exists K \in \mathbb{R}^{n\times n} \mid \frac{\partial F(x)}{\partial x} + K$ *is sign-stable over* $[\underline{x}, \overline{x}]$

Theorem

$\forall i \in \{1, \dots, n\}$, *there exist states* $\underline{\xi}^i = \begin{pmatrix} \underline{\xi}^i_1 \\ \vdots \\ \underline{\xi}^i_n \end{pmatrix}$ *and* $\overline{\xi}^i = \begin{pmatrix} \overline{\xi}^i_1 \\ \vdots \\ \overline{\xi}^i_n \end{pmatrix}$ *such that*

$$F_i([\underline{x}, \overline{x}]) \subseteq [F_i(\underline{\xi}^i) - K_{i*}(\overline{\xi}^i - \underline{\xi}^i), F_i(\overline{\xi}^i) + K_{i*}(\overline{\xi}^i - \underline{\xi}^i)]$$

$$(\underline{\xi}^i_j, \overline{\xi}^i_j) = \begin{cases} (\underline{x}_j, \overline{x}_j) \text{ if } \dfrac{\partial F_i(x)}{\partial x_j} + K_{ij} \geq 0 \\ (\overline{x}_j, \underline{x}_j) \text{ if } \dfrac{\partial F_i(x)}{\partial x_j} + K_{ij} \leq 0 \end{cases} \qquad K_{i*} = \begin{pmatrix} K_{i1} & \dots & K_{in} \end{pmatrix}$$

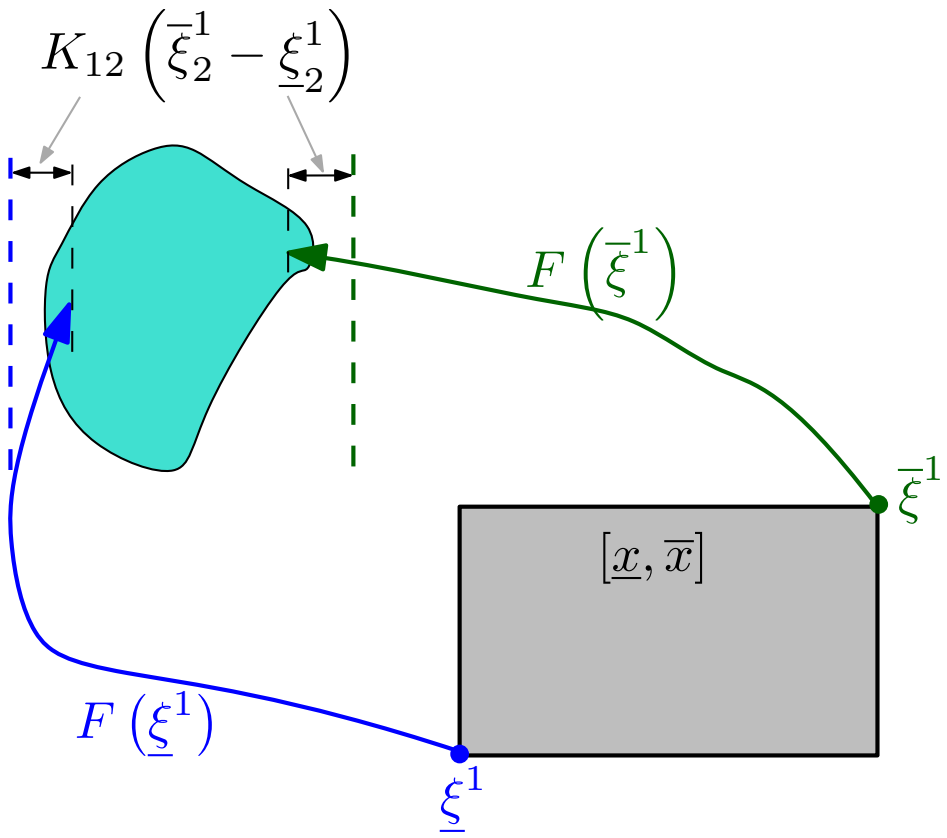
Bounded Jacobian - Illustration

$$sign\left(\frac{\partial F(x)}{\partial x} + K\right) = \begin{pmatrix} + & + \\ - & + \end{pmatrix} \qquad K = \begin{pmatrix} 0 & K_{12} \\ 0 & 0 \end{pmatrix}$$

Dimension 1:

$$\underline{\xi}^1 = \begin{pmatrix} \underline{x}_1 \\ \underline{x}_2 \end{pmatrix} \quad \overline{\xi}^1 = \begin{pmatrix} \overline{x}_1 \\ \overline{x}_2 \end{pmatrix}$$

Correction: $\pm K_{12}\left(\overline{\xi}_2^1 - \underline{\xi}_2^1\right)$



Bounded Jacobian - Illustration

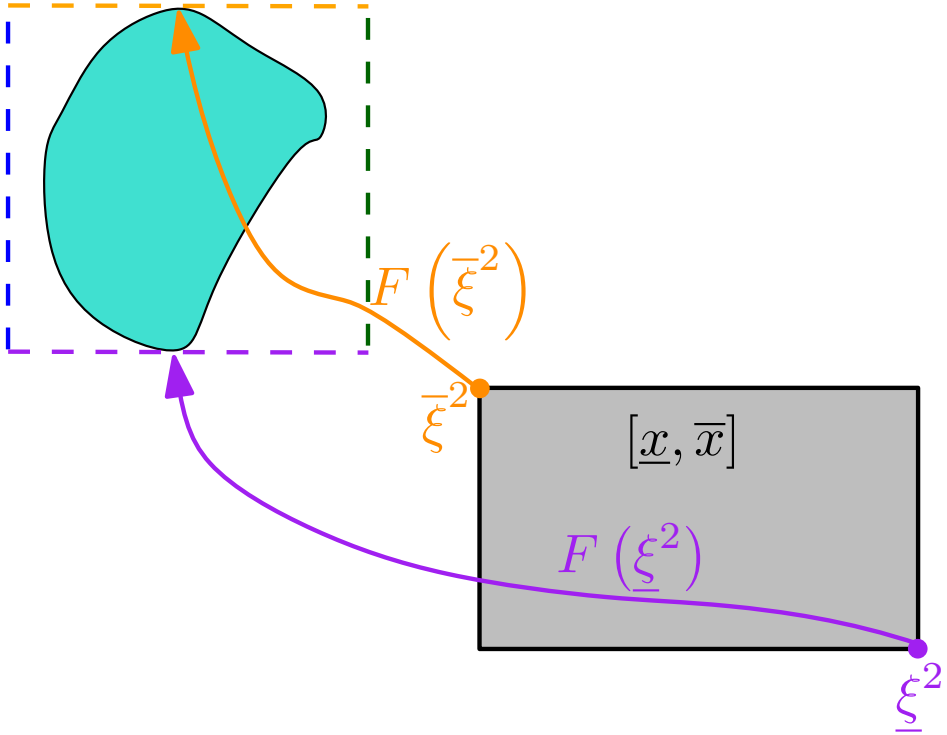
$$sign\left(\frac{\partial F(x)}{\partial x} + K\right) = \begin{pmatrix} + & + \\ - & + \end{pmatrix} \qquad K = \begin{pmatrix} 0 & K_{12} \\ 0 & 0 \end{pmatrix}$$

Dimension 2:

$$\underline{\xi}^2 = \begin{pmatrix} \overline{x}_1 \\ \underline{x}_2 \end{pmatrix} \quad \overline{\xi}^2 = \begin{pmatrix} \underline{x}_1 \\ \overline{x}_2 \end{pmatrix}$$

No correction: $K_{2*} = \begin{pmatrix} 0 & 0 \end{pmatrix}$

$F_2([\underline{x}, \overline{x}]) \subseteq [F_2(\underline{\xi}^2), F_2(\overline{\xi}^2)]$



Lose tightness (unless K has some rows of 0)
Keep linear complexity ($2n$ evaluations of F)

Continuous-time case

$$\dot{x} = f(x), \quad x(t) = \Phi(t; x_0), \quad RS(T, [\underline{x}, \overline{x}]) = \{\Phi(T; x_0) \mid x_0 \in [\underline{x}, \overline{x}]\}$$

Sampled-data system: $x^+ = F(x) = \Phi(T; x)$

Continuous-time case

$$\dot{x} = f(x), \quad x(t) = \Phi(t; x_0), \quad RS(T, [\underline{x}, \overline{x}]) = \{\Phi(T; x_0) \mid x_0 \in [\underline{x}, \overline{x}]\}$$

Sampled-data system: $x^+ = F(x) = \Phi(T; x)$

Assumption (Bounded sensitivity matrix)

$\frac{\partial \Phi(T;x_0)}{\partial x_0}$ *is* **bounded**: $\exists K \in \mathbb{R}^{n \times n} \mid \frac{\partial \Phi(T;x_0)}{\partial x_0} + K$ *is sign-stable over* $[\underline{x}, \overline{x}]$

Theorem

$\forall i \in \{1, \ldots, n\}$, *there exist states* $\underline{\xi}^i, \overline{\xi}^i \in [\underline{x}, \overline{x}]$ *such that*

$$RS_i(T, [\underline{x}, \overline{x}]) \subseteq [\Phi_i(T; \underline{\xi}^i) - K_{i*}(\overline{\xi}^i - \underline{\xi}^i), \Phi_i(T; \overline{\xi}^i) + K_{i*}(\overline{\xi}^i - \underline{\xi}^i)]$$

$$(\underline{\xi}^i_j, \overline{\xi}^i_j) = \begin{cases} (\underline{x}_j, \overline{x}_j) \text{ if } \dfrac{\partial \Phi_i(T; x)}{\partial x_j} + K_{ij} \geq 0 \\ (\overline{x}_j, \underline{x}_j) \text{ if } \dfrac{\partial \Phi_i(T; x)}{\partial x_j} + K_{ij} \leq 0 \end{cases} \qquad K_{i*} = \begin{pmatrix} K_{i1} & \ldots & K_{in} \end{pmatrix}$$

Obtaining sensitivity bounds

$$s(t;x_0) = \frac{\partial \Phi(t;x_0)}{\partial x_0}, \qquad \dot{s}(t;x_0) = \left.\frac{\partial f(x)}{\partial x}\right|_{\Phi(t;x_0)} s(t;x_0) \tag{1}$$

Interval arithmetics[1], using bounds on the Jacobian $\left.\frac{\partial f(x)}{\partial x}\right|_{\Phi(t;x_0)}$

- guaranteed to over-approximate the set $\{s(T;x_0) \mid x_0 \in [\underline{x}, \overline{x}]\}$
- overly conservative bounds

Simulation-based estimation: sampling and falsification

- numerical integration of (1) for **sampled** initial states in $[\underline{x}, \overline{x}]$
- optimization to find $x_0 \in [\underline{x}, \overline{x}]$ **falsifying** the current bounds
- not guaranteed to cover the set $\{s(T;x_0) \mid x_0 \in [\underline{x}, \overline{x}]\}$

[1] M. Althoff, O. Stursberg, and M. Buss. *Reachability analysis of linear systems with uncertain parameters and inputs*. CDC07.

Time and parameter uncertainty

Uncertain time-varying system: $\dot{x}(t) = f(t, x(t), p), \quad p \in [\underline{p}, \overline{p}]$

$$RS(T; t_0, [\underline{x}, \overline{x}], [\underline{p}, \overline{p}]) = \{\Phi(T; t_0, x_0, p) \mid x_0 \in [\underline{x}, \overline{x}],\ p \in [\underline{p}, \overline{p}]\}$$

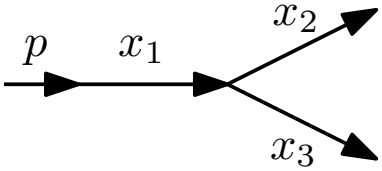
Assumption

Both sensitivities $\dfrac{\partial \Phi(T; t_0, x_0, p)}{\partial x_0}$ *and* $\dfrac{\partial \Phi(T; t_0, x_0, p)}{\partial p}$ *are bounded*

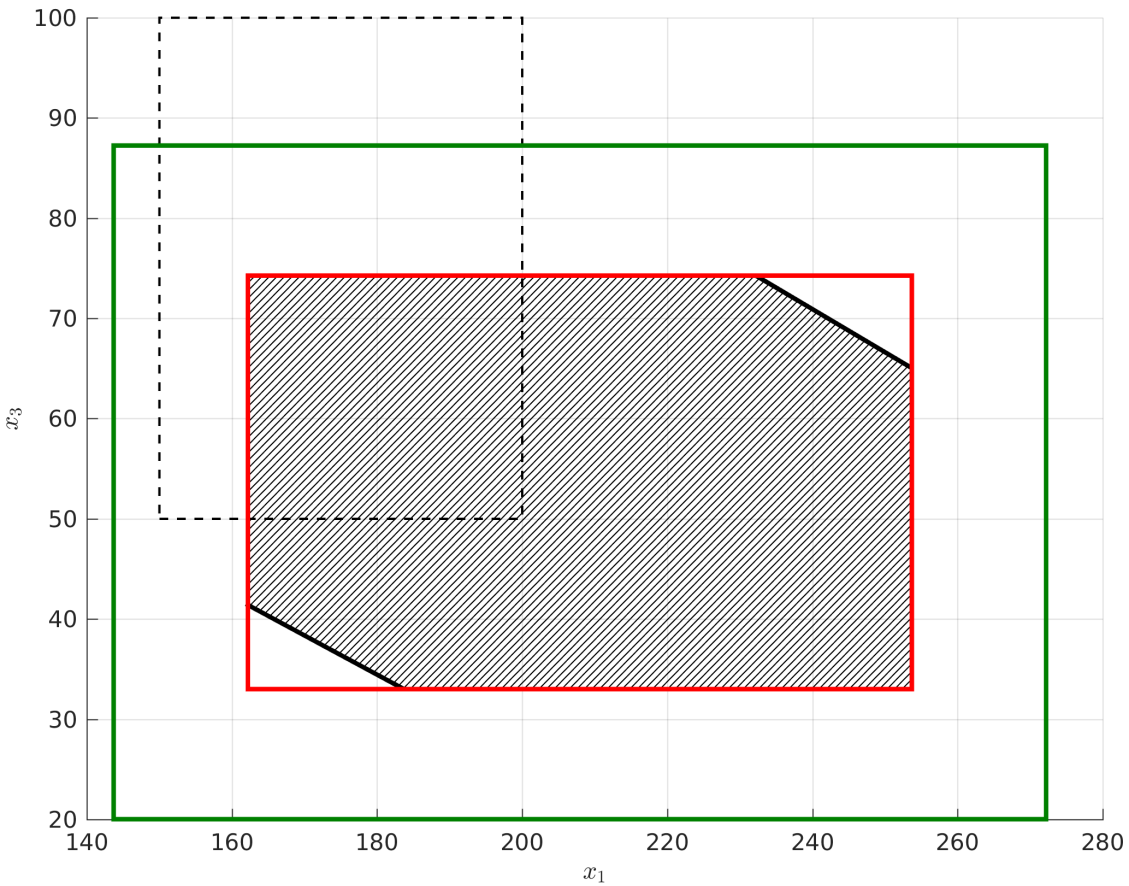
Theorem

$$RS_i(T; t_0, [\underline{x_0}, \overline{x_0}], [\underline{p}, \overline{p}]) \subseteq$$
$$[\Phi_i(T; t_0, \underline{\xi}^i, \underline{\pi}^i) - K_{i*}(\overline{\xi}^i - \underline{\xi}^i) - L_{i*}(\overline{\pi}^i - \underline{\pi}^i),$$
$$\Phi_i(T; t_0, \overline{\xi}^i, \overline{\pi}^i) + K_{i*}(\overline{\xi}^i - \underline{\xi}^i) + L_{i*}(\overline{\pi}^i - \underline{\pi}^i)]$$

Numerical example: traffic network



Piecewise affine model:
$\dot{x} = f(x, p)$
$x \in \mathbb{R}^3$: link densities
p: constant input to link 1

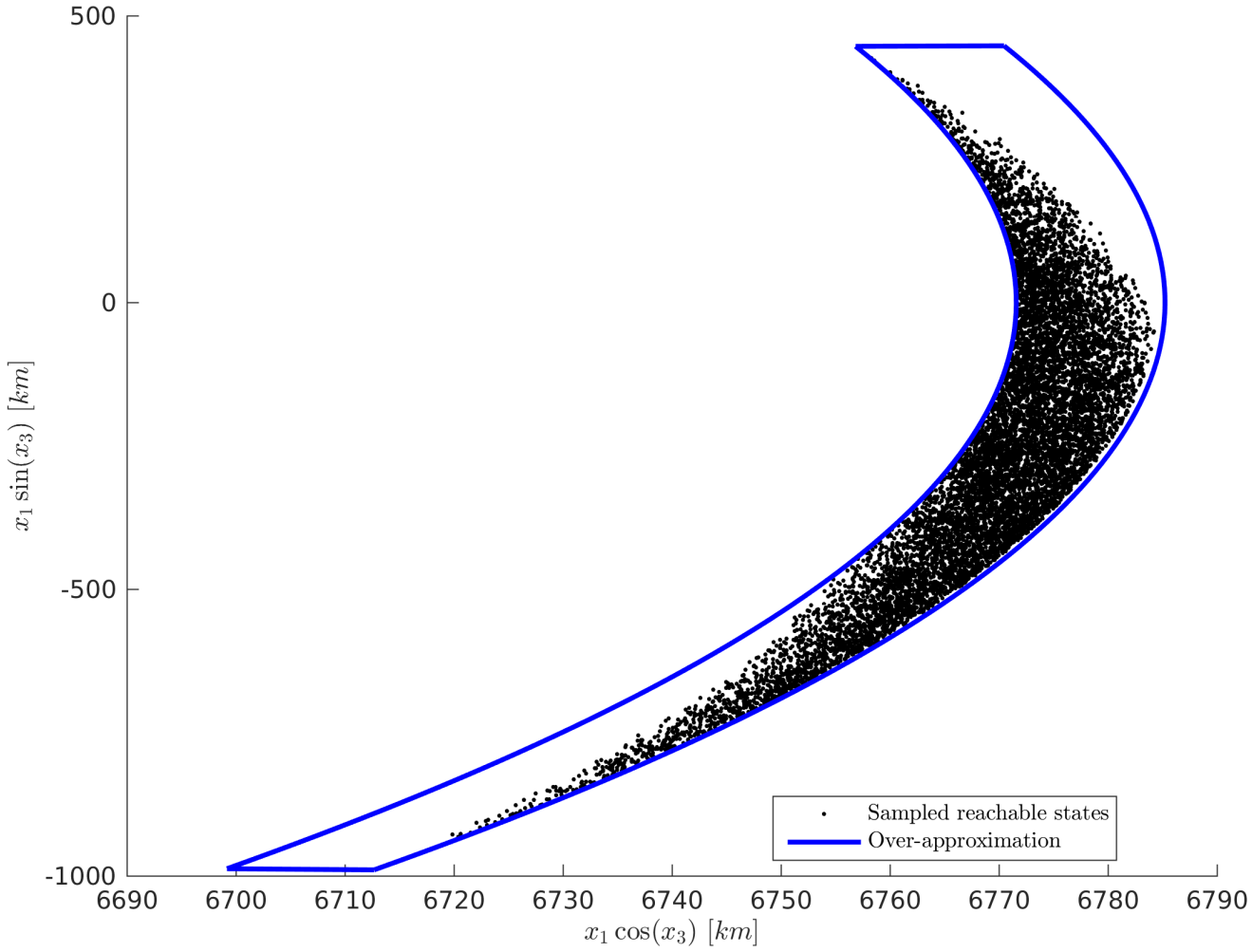


Sampling/falsification (7 s)
Interval arithmetics (14 ms)

Numerical example: satellite orbit

$$\dot{x} = \begin{pmatrix} x_2 \\ -\frac{p}{x_1^2} + x_1 x_4^2 \\ x_4 \\ -\frac{2x_2x_4}{x_1} \end{pmatrix}$$

Over-approximation interval at $T = 92$ minutes in the **polar coordinates**:



Conclusions

Interval over-approximation of the reachable set

- **for any time-varying system with constant inputs**
- both discrete-time and continuous-time
- **linear complexity** in the state dimension

Continuous-time systems: 2 methods to obtain sensitivity bounds

- **Interval arithmetics**: guaranteed bounds but conservative
- **Sampling and falsification**: simulation-based

Future direction

- Improve quality/complexity tradeoff of sensitivity bound methods
- TIRA: Toolbox for Interval Reachability Analysis [2]

[2] `https://gitlab.com/pj_meyer/TIRA`