# Abstraction Refinement for Control Synthesis: A Discrete-Time Hybridization Approach

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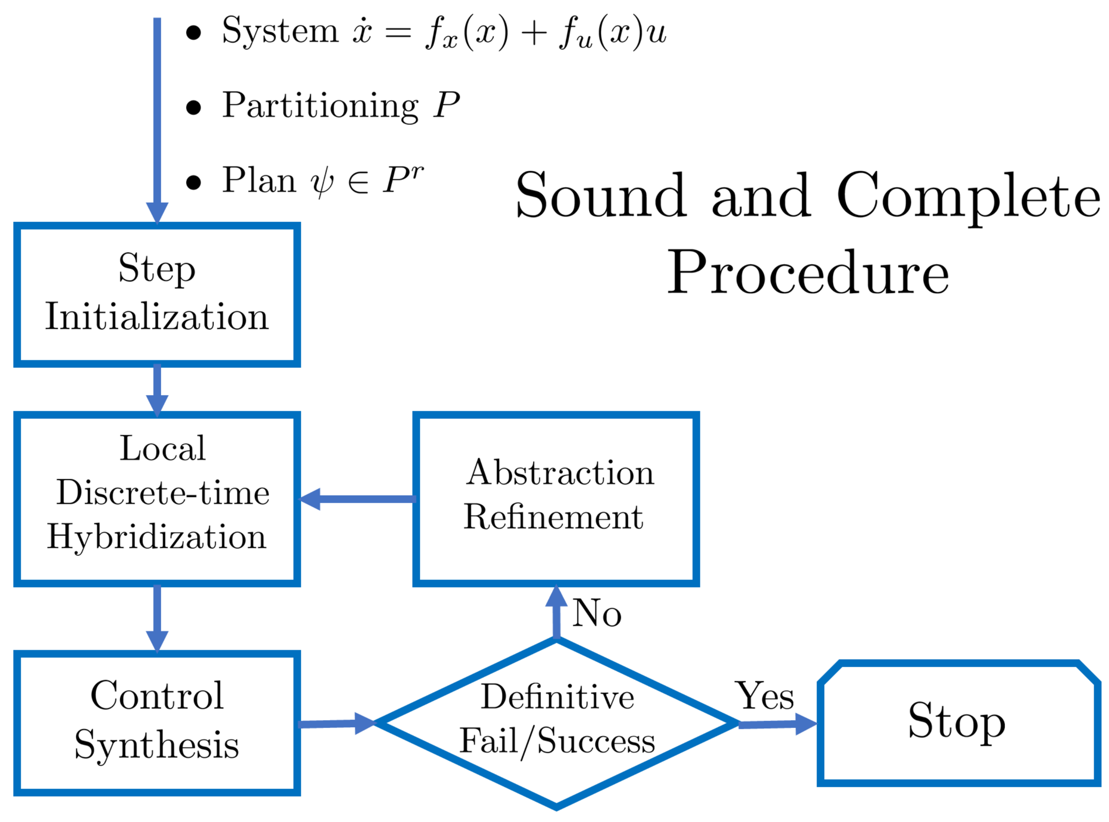
**Problem:** Synthesize a controller for a nonlinear system to follow a sequence of regions in the partitioned state space.

**Approach:**

1. *Discrete-time Hybridization*: the sampled nonlinear dynamics restricted to each considered partition cell over-approximated by an affine abstraction.
2. *Control Synthesis*: control synthesis problem and relaxed control synthesis problem (feasibility verification) as a Mixed-integer linear programming (MILP).
3. *Abstraction-Refinement*: refining the abstraction into a piecewise affine model in case of inconclusive result.




## Discrete-time hybridization

Derivation of a discrete-time (with time step $\tau$) affine model

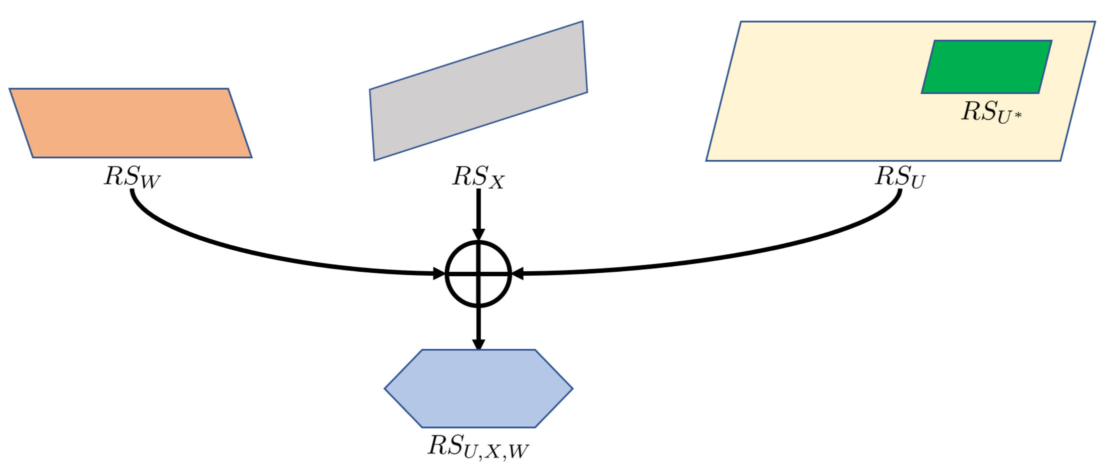
$$x^+ = Ax + Bu + c + \boldsymbol{W}$$

of a system $\dot{x} = f_x(x) + f_u(x)u$ that is valid within a box region $X$:

$$\forall x \in X,\ \phi(x, u, \tau) \in Ax + Bu + c + \boldsymbol{W}.$$

**Challenges and solutions:**

- $\dot{x} = f_x(x) + f_u(x)u$ is *not in general discretizable* therefore we compute an affine over-approximation of the continuous dynamics and then discretize.
- The state *will not remain inside* $X$ during time interval $\tau$ therefore the continuous time over-approximation has to be valid over the *reachable set*. We iteratively compute an approximation/reachable set until reachable set falls within the area where approximation is valid.
- To ensure completness, we *limit the input range* using *zonotope representation*

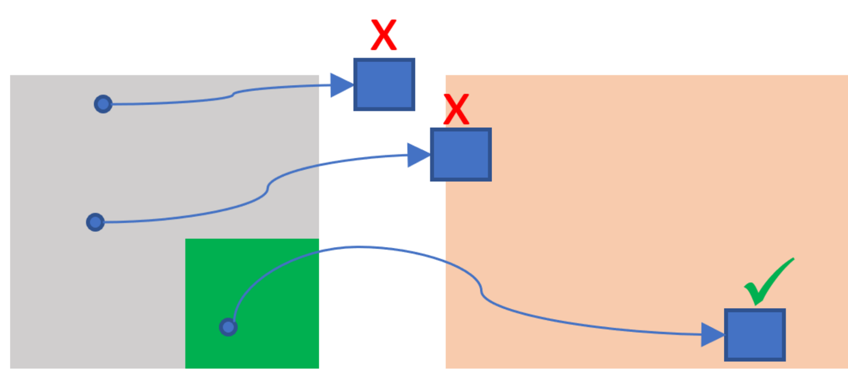



$$\forall u \in U^*,\ \exists (x', w') \in (RS_X, RS_W) \text{ s.t. } RS_{x,u,w} \in \zeta$$

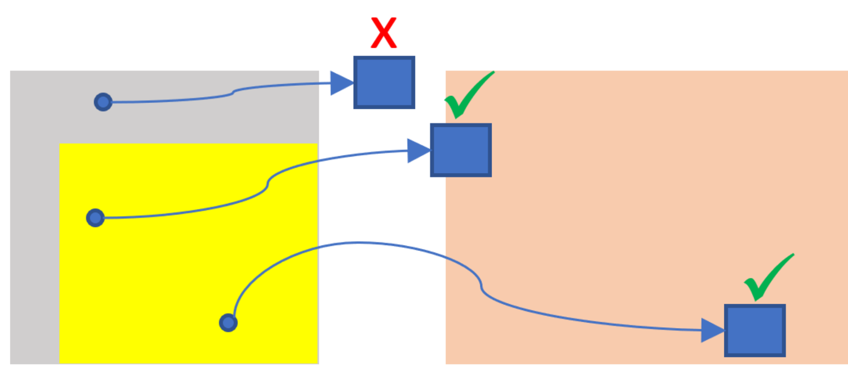
## Control Synthesis

The control synthesis is formulated as two mixed-integer linear programming problems:

$$\begin{aligned} \max_{\mu,d,\mathcal{C}}\ & size(B(\mu,d)) \\ \text{s.t.}\ & B(\mu,d) \subseteq \theta \\ & \forall x \in B(\mu,d),\ \mathcal{C}(x) \subseteq \mathcal{U} \\ & \forall x \in B(\mu,d),\ \overline{RS}_{\mathcal{C}}(x) \subseteq \zeta \end{aligned}$$



$$\begin{aligned} \max_{\mu,d,\mathcal{C}}\ & size(B(\mu,d)) \\ \text{s.t.}\ & B(\mu,d) \subseteq \theta \\ & \forall x \in B(\mu,d),\ \mathcal{C}(x) \subseteq \mathcal{U} \\ & \forall x \in B(\mu,d),\ \overline{RS}_{\mathcal{C}}(x) \cap \zeta \neq \emptyset \end{aligned}$$



## Abstraction Refinement

If problem 1 fails but 2 succeeds, the gridding of the state space has to be refined as the failure may be a result of *spurious behaviour* introduced by the approximation.


## Results

**System*:**

$$\dot{x} = \begin{bmatrix} -1 & 0.3 \\ 0.3 & -1 \end{bmatrix} x - 0.01x^3 + u$$

$$u \in [-5, 5]^2$$

* Adopted from Meyer and Dimarogonas (2017), "Abstraction refinement and plan revision for control synthesis under high level specifications" for comparison of results: Larger valid sets and no monotonocity requirement.