# TIRA: Toolbox for Interval Reachability Analysis

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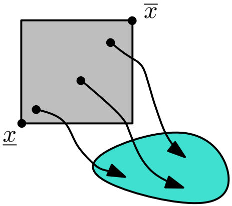
# Reachability analysis

**Discrete-time system**
$x^+ = F(t, x, p)$

**Reachability problem**

- Initial time: $t_0$
- Initial states $[\underline{x}, \overline{x}]$
- Input bounds $[\underline{p}, \overline{p}]$

**Reachable set** in one discrete step
$\{F(t_0, x, p) | x \in [\underline{x}, \overline{x}], p \in [\underline{p}, \overline{p}]\}$



# Reachability analysis

**Discrete-time system**
$x^+ = F(t, x, p)$

**Reachability problem**

- Initial time: $t_0$
- Initial states $[\underline{x}, \overline{x}]$
- Input bounds $[\underline{p}, \overline{p}]$

**Reachable set** in one discrete step
$\{F(t_0, x, p) | x \in [\underline{x}, \overline{x}], p \in [\underline{p}, \overline{p}]\}$

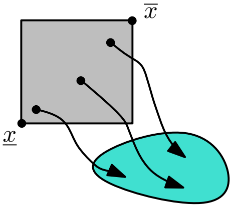
**Continuous-time system**
$\dot{x} = f(t, x, p)$

**Reachability problem**

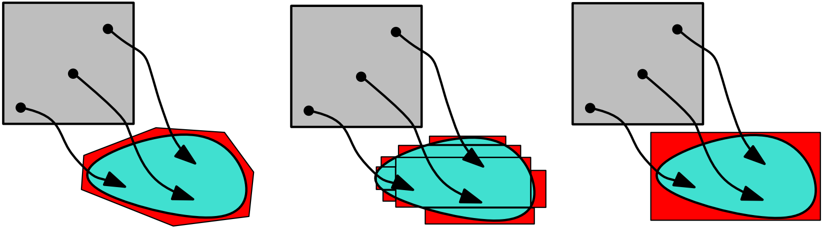
- Time range: $[t_0, t_f]$
- Initial states $[\underline{x}, \overline{x}]$
- Input bounds $[\underline{p}, \overline{p}]$

**Reachable set** at final time $t_f$

$$\{x(t_f; t_0, x_0, p) | x_0 \in [\underline{x}, \overline{x}], \quad p : [t_0, t_f] \to [\underline{p}, \overline{p}]\}$$



# Over-approximations



**Flexible set representations**

- polytopes
- ellipsoids
- level sets
- interval pavings

**Accurate approximations**
**Low scalability**

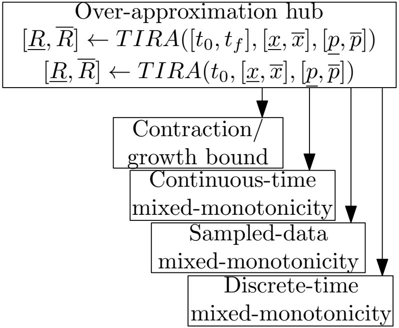
**Intervals**

- easy to manipulate
- defined with only 2 state vectors
- intersection is still an interval

**Good scalability**
**Lower accuracy**

# Architecture

## Core architecture

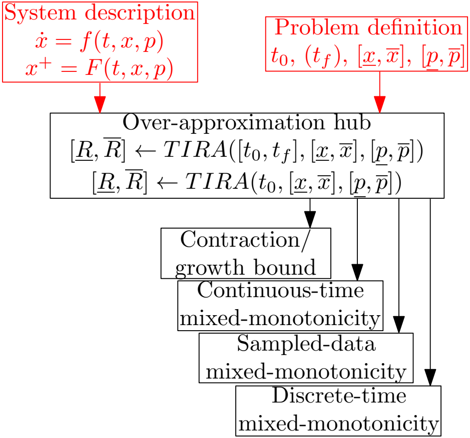


# Architecture

Core architecture

**User-inputs**

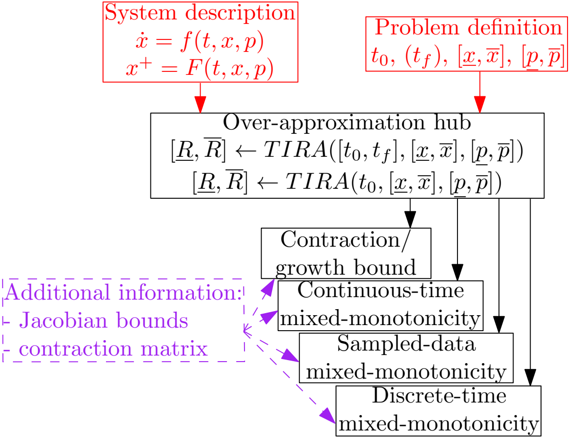
- **Required**



Core architecture

**User-inputs**

- Required
- **Recommended**

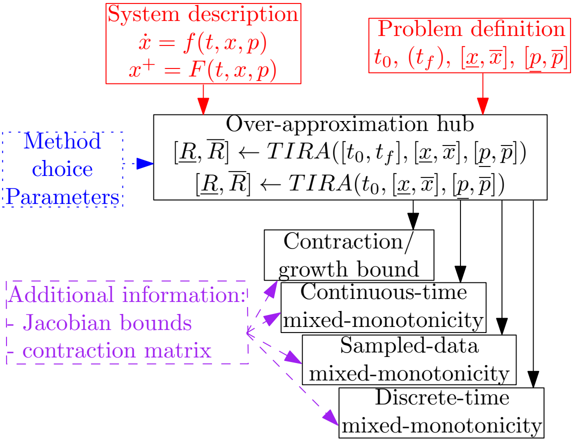


# Architecture

Core architecture

**User-inputs**

- Required
- Recommended
- **Optional**



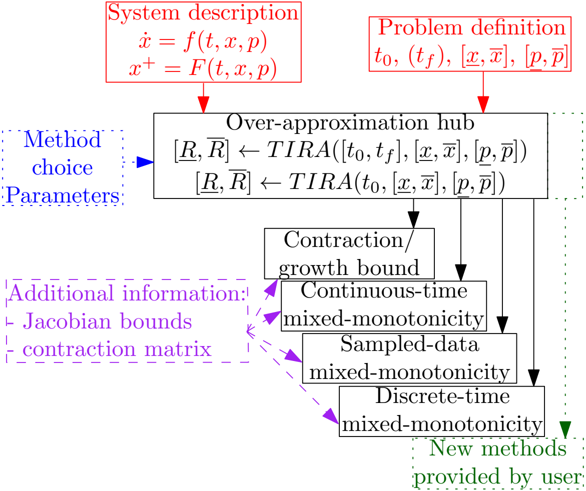
# Architecture

Core architecture

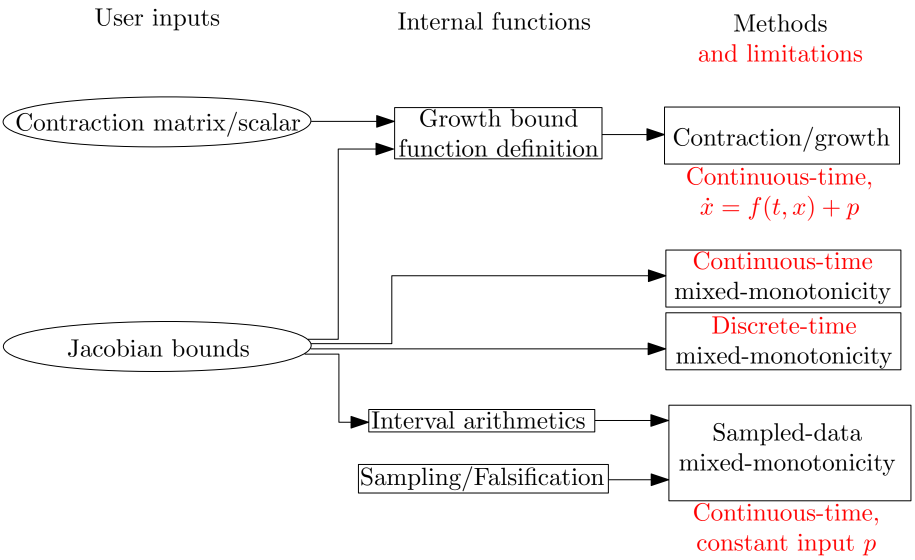
User-inputs

- Required
- Recommended
- Optional

**Extensible architecture**



# Over-approximation methods



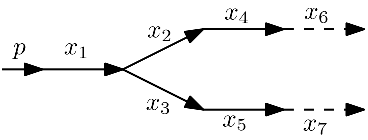
# Method comparison

State $x \in \mathbb{R}^n$

| **Method** | **Tightness conditions** | **Complexity** |
|---|---|---|
| Contraction/growth bound | | 1 |
| CT mixed-monotonicity | monotone | 2 |
| DT mixed-monotonicity | sign-stable Jacobian | $2n$ |
| SD mixed-monotonicity (IA) | sign-stable sensitivity | $\geq O(2n)$ |
| SD mixed-monotonicity (SF) | sign-stable sensitivity | $O(2^n)$ |

Complexity $\sim$ number of successor evaluations

# Illustration: traffic network

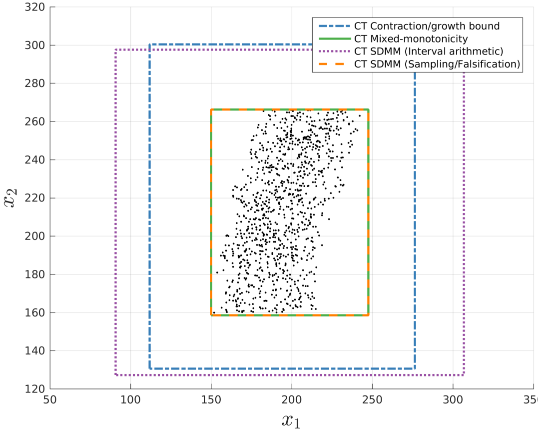



Piecewise affine model:

$\dot{x} = f(x, p)$

$x \in \mathbb{R}^n$: road section densities

$p$: constant input to section 1




| **Method** | $n = 3$ | $n = 99$ |
|---|---|---|
| Contraction/growth bound | 0.13 s | 0.37 s |
| CT mixed-monotonicity | 0.05 s | 4.4 s |
| SDMM (Interval arithmetics) | 0.28 s | 338 s |
| SDMM (Sampling/falsification) | 7 s | — |

# Conclusions

**Matlab library of 4 interval reachability methods**

- covering most nonlinear systems
- both continuous-time and discrete-time systems
- good scalability

**Toolbox architecture**

- easily extensible
- can choose most suitable method

`https://gitlab.com/pj_meyer/TIRA`
`pjmeyer@berkeley.edu`

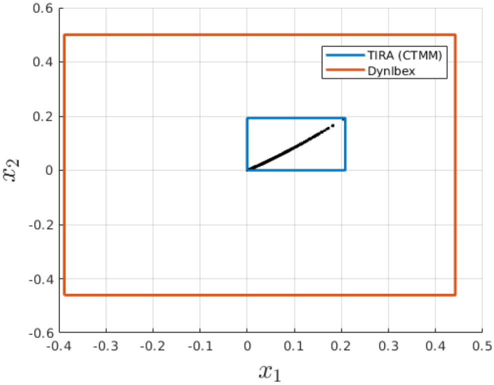
# Comparison with DynIbex[1]

**Protein interactions**[2]

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} -x_1 + x_2 \\ \frac{x_1^2}{1+x_2^2} - 0.3x_2 \end{pmatrix}$$

- Initial set: $[0, 0.3]^2$
- Time range: $[0, 10]$ s

| **Method** | **Time** |
|---|---|
| TIRA (CTMM) | 26 ms |
| DynIbex | 1.9 s |




[1] Julien Alexandre and Alexandre Chapoutot, *Validated explicit and implicit Runge-Kutta methods*. Reliable Computing v. 22, 2016.

[2] Lee A. Segel, *Modeling dynamic phenomena in molecular and cellular biology*. Cambridge University Press, 1984.