Interval Reachability Analysis using Second-Order Sensitivity

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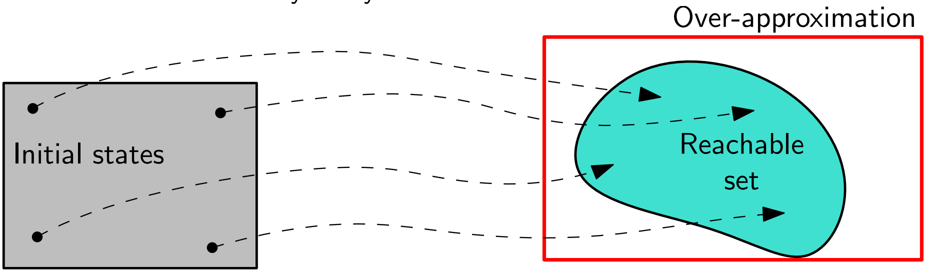
Reachability analysis

Continuous-time system: $\dot{x} = f(x)$

Objective: finite-time reachability analysis from initial interval



Exact computation of the reachable set: impossible

$\rightarrow$ **over-approximation** by a **multi-dimensional interval**

Motivations

	Discrete-time mixed-monotonicity [1]	Sampled-data mixed-monotonicity
System	$x^+ = F(x)$	$\dot{x} = f(x)$ $x^+ = x(T; x_0)$
Requirement	Bounded Jacobian $\dfrac{\partial F(x)}{\partial x} \in [\underline{J}, \overline{J}]$	Bounded sensitivity $\dfrac{\partial x(T; x_0)}{\partial x_0} \in [\underline{S}, \overline{S}]$

Main challenge: compute sensitivity bounds with a tunable complexity/accuracy tradeoff

[1] Meyer, Coogan and Arcak, *IEEE Control Systems Letters*, 2018

Jacobian and Sensitivity definitions

Jacobian matrices of the continuous-time system $\dot{x} = f(x)$:

$$J^x(x) = \frac{\partial f(x)}{\partial x} \qquad J^{xx}(x) = \frac{\partial J^x(x)}{\partial x}$$

First-order sensitivity

$$S^x(t; x_0) = \frac{\partial x(t; x_0)}{\partial x_0}$$

$$\dot{S}^x = J^x * S^x$$
$$S^x(0; x_0) = I_n$$

Second-order sensitivity

$$S^{xx}(t; x_0) = \frac{\partial S^x(t; x_0)}{\partial x_0}$$

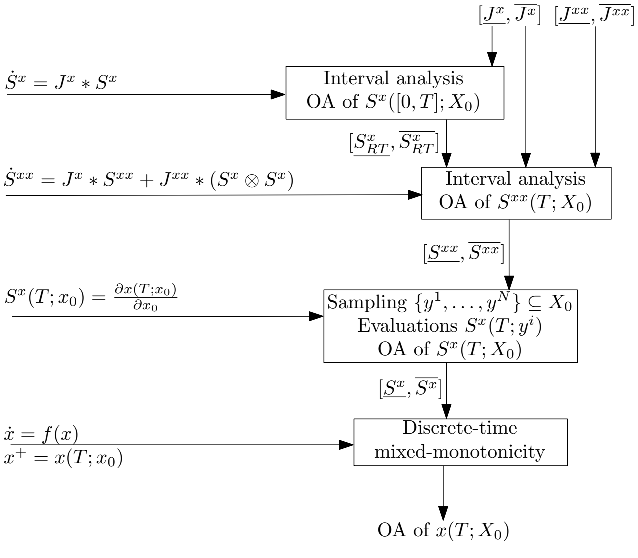
$$\dot{S}^{xx} = J^x * S^{xx} + J^{xx} * (S^x \otimes S^x)$$
$$S^{xx}(0; x_0) = 0$$

Overview of the 4-step reachability procedure

3 steps to bound S^x

- Interval analysis[2] on linear systems of S^x and S^{xx}
- Grid-based sampling on S^x

Last step for discrete-time mixed-monotonicity on sampled-data system



[2] Althoff, Stursberg and Buss, CDC 2007

Step 1: First-order sensitivity tube

Linear system $\dot{S}^x = J^x * S^x$ with initial condition $S^x(0) = I_n$

- If J^x is constant, solutions are known: $S^x(t) = e^{J^x * t}$
- If $J^x \in [\underline{J^x}, \overline{J^x}]$: $S^x(t) \in e^{[\underline{J^x}, \overline{J^x}] * t}$
- *Interval matrix exponential* evaluated using Taylor expansion and interval arithemtics:

$$e^{[\underline{J^x}, \overline{J^x}] * t} \subseteq \sum_{i=0}^{+\infty} \frac{([\underline{J^x}, \overline{J^x}] * t)^i}{i!}$$

- Truncate Taylor expansion and over-approximate remainder

Reachable tube $S^x([0, T])$

- Interval hull of initial $S^x(0) = I_n$ and final solutions $S^x(T) \in e^{[\underline{J^x}, \overline{J^x}] * T}$
- Enlarge hull to guarantee over-approximation

Step 2: Second-order sensitivity set

Affine system $\dot{S}^{xx} = J^x * S^{xx} + J^{xx} * (S^x \otimes S^x)$ with initial condition $S^{xx}(0) = 0$

- Similar approach defining an interval affine system
- Need bounds on J^x and $J^{xx} * (S^x \otimes S^x)$ for all $t \in [0, T]$
  - Bounds on J^x and J^{xx} assumed to be provided
  - This is why step 1 computed the reachable **tube** of S^x
- Denote $J^{xx} * (S^x \otimes S^x) \in [\underline{B}, \overline{B}]$

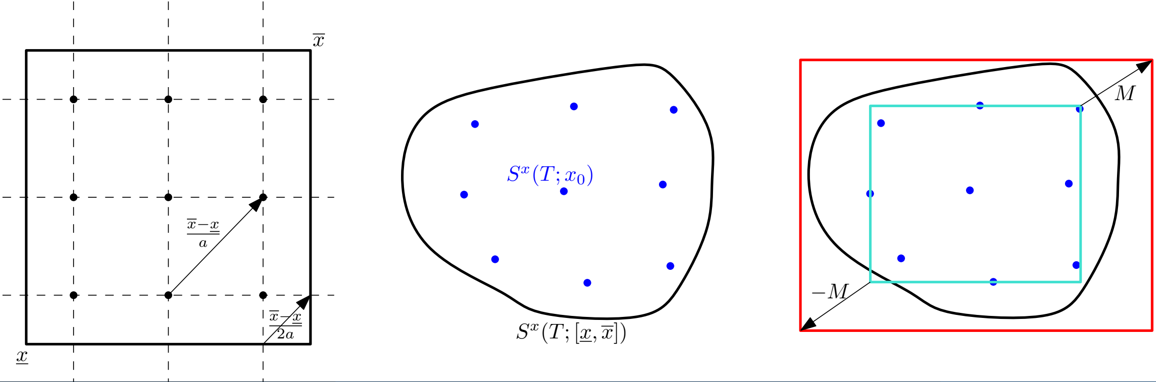
Second-order sensitivity **reachable set**:

$$S^{xx}(T) \in \int_0^T e^{[\underline{J^x}, \overline{J^x}]*t} dt * [\underline{B}, \overline{B}]$$

Step 3: First-order sensitivity set

Guaranteed over-approximation of $S^x(T; [\underline{x}, \overline{x}])$ from sampling

- Uniform grid sampling of interval of initial states (a samples per dimension)
- Evaluation of $S^x(T; x_0)$ for each sample x_0
- Interval hull of sampled sensitivity evaluations
- Expand hull by $M = \max\left(|\underline{S^{xx}}|, |\overline{S^{xx}}|\right) * \left(I_n \otimes \left(\mathbf{1}_n * \frac{\|\overline{x}-\underline{x}\|_\infty}{2a}\right)\right)$



Step 4: Reachable set of the continuous-time system

Discrete-time mixed-monotonicity [2]

- applied to sampled-data system $x^+ = x(T; x_0)$
- using first-order sensitivity bounds $S^x(T; x_0) \in [\underline{S^x}, \overline{S^x}]$ (centered on S^{x*})

Auxiliary function $g : X \times X \to X$

- $g_i(x, y) = x_i(T; z^i) + \alpha^i(x - y)$
- with state $z^i = [z^i_1; \dots; z^i_n] \in \mathbb{R}^n$ and row vector $\alpha^i = [\alpha^i_1, \dots, \alpha^i_n] \in \mathbb{R}^{1\times n}$

$$(z^i_j, \alpha^i_j) = \begin{cases} (x_j, \max(0, -\underline{S^x}_{ij})) & \text{if } S^{x*}_{ij} \geq 0, \\ (y_j, \max(0, \overline{S^x}_{ij})) & \text{if } S^{x*}_{ij} < 0. \end{cases}$$

Theorem (Final reachable set)

$$x(T; [\underline{x}, \overline{x}]) \subseteq [g(\underline{x}, \overline{x}), g(\overline{x}, \underline{x})]$$

[2] Meyer, Coogan and Arcak, *IEEE Control Systems Letters*, 2018

Comparisons

Alternative 1-step approaches [3] to bound $S^x(T, [\underline{x}, \overline{x}])$

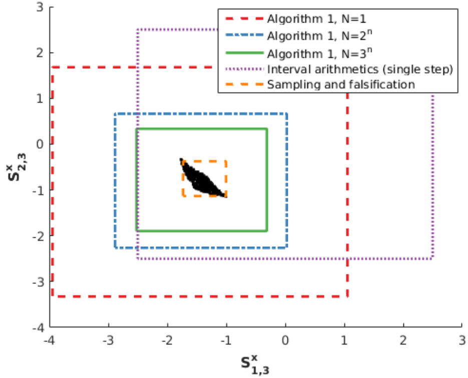
- Interval analysis on $\dot{S}^x = J^x * S^x$
- Random sampling of $S^x(T, x_0)$ without bounds on S^{xx}

	Interval analysis	Sampling	3-step approach
OA guarantees	**yes**	no	**yes**
Conservativeness	large	**small**	**tunable**
Complexity	**low**	high	**tunable**
Requirements	$[\underline{J^x}, \overline{J^x}]$	**none**	$[\underline{J^x}, \overline{J^x}]$, $[\underline{J^{xx}}, \overline{J^{xx}}]$

[3] Meyer, Coogan and Arcak, *IEEE Control Systems Letters*, 2018

Unicycle with constant uncertainties

$$\dot{x} = \begin{pmatrix} v\cos(x_3) + x_4 \\ v\sin(x_3) + x_5 \\ \omega + x_6 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$



	Interval analysis	Random sampling	3-step approach		
Samples N	-	64	1	64	729
Computation time for $[\underline{S^x}, \overline{S^x}]$	0.44 s	3.1 s	0.35 s	3.2 s	36 s

Conclusions

- Interval over-approximation for the reachable set of continuous-time systems
- Based on discrete-time mixed-monotonicity
- 3-step bounding of sensitivity matrix, with **tunable complexity/conservativeness**

Future work

- Include this new method to the Matlab toolbox TIRA[4]

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[4] Toolbox for Interval Reachability Analysis: `https://gitlab.com/pj_meyer/TIRA`